CALCULATING THE THERMOELASTIC EQUILIBRIUM OF A TOROIDAL SHELL

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UDC 624.074.4

In [1, 2] an algorithm is proposed for the calculation of Green's matrices which easily can be extended to the case of closed shells of revolution, in particular, spherical, toroidal, and others. The use of these matrices effectively allows us to determine the stress-strain state of such shells, generally speaking, for arbitrary loading. On the other hand, the calculation of the stress state of a nonuniformly heated shell usually leads to the consideration of the so-called temperature loads of fairly complex structure. Below, in certain examples which have important practical significance, we investigate the possibility of setting up algorithms of such calculation which are based on the use of Green's matrices calculated beforehand. The numerical results thus obtained allow us to draw certain conclusions.

Let the parametric form of the toroidal surface in question be given by the equations

$$x = (R + a\cos\varphi)\cos\vartheta,\quad y = (R + a\cos\varphi)\sin\vartheta,\quad z = a\sin\varphi \tag{1}$$

The thermoelastic equilibrium for which this surface is the middle surface (as a shell of revolution) can be determined ([3], p. 98) by means of a system of differential equations of the form

$$\left[A^\circ\left(\varphi, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\right) + \frac{h^2}{12} A^*\left(\varphi, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\right)\right] U(\varphi, \vartheta) = \Theta(\varphi, \vartheta)$$

$$U(\varphi, \vartheta) = \begin{pmatrix} u(\varphi, \vartheta) \\ v(\varphi, \vartheta) \\ w(\varphi, \vartheta) \end{pmatrix}, \qquad \Theta(\varphi, \vartheta) = \begin{pmatrix} \Theta_1(\varphi, \vartheta) \\ \Theta_2(\varphi, \vartheta) \\ \Theta_3(\varphi, \vartheta) \end{pmatrix} \tag{2}$$

where $U(\varphi, \vartheta)$ and $\Theta(\varphi, \vartheta)$ are the vectors of the displacement of the middle surface and the temperature load respectively, and the operator A° has the form

$$A^\circ = (A_{ij}{}^\circ)_{i,j=1}^3$$

$$A_{11}{}^\circ = \frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1-\nu}{2B^2}\frac{\partial^2}{\partial\vartheta^2} - \frac{\sin\varphi}{aB}\frac{\partial}{\partial\varphi} + \frac{(1-\nu)\cos\varphi}{aB} - \frac{a + R\cos\varphi}{aB^2}$$

$$A_{12}{}^\circ = \frac{1+\nu}{2aB}\frac{\partial^2}{\partial\varphi\,\partial\vartheta} + \frac{(3-\nu)\sin\varphi}{2B^2}\frac{\partial}{\partial\vartheta}$$

$$A_{13}{}^\circ = \frac{R + (1+\nu)a\cos\varphi}{a^2B}\frac{\partial}{\partial\varphi} - \frac{R\sin\varphi}{aB^2}$$

$$A_{21}{}^\circ = \frac{1+\nu}{2aB}\frac{\partial^2}{\partial\varphi\,\partial\vartheta} - \frac{(3-\nu)\sin\varphi}{2B^2}\frac{\partial}{\partial\vartheta}$$

$$A_{22}{}^\circ = \frac{1}{B^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{1-\nu}{2a^2}\frac{\partial^2}{\partial\varphi^2} - \frac{(1-\nu)\sin\varphi}{2aB}\frac{\partial}{\partial\varphi} + (1-\nu)\frac{(R+B)\cos\varphi - a\sin^2\varphi}{2aB^2}$$

$$A_{23}{}^\circ = \frac{R + (1+\nu)a\cos\varphi}{aB^2}\frac{\partial}{\partial\vartheta}$$

$$A_{31}{}^\circ = \frac{R + (1+\nu)a\cos\varphi}{a^2B}\frac{\partial}{\partial\varphi} - \frac{[\nu R + (1+\nu)a\cos\varphi]\sin\varphi}{aB^2}$$

$$A_{32}{}^\circ = \frac{\nu R + (1+\nu)a\cos\varphi}{aB^2}\frac{\partial}{\partial\vartheta}$$

$$A_{33}{}^\circ = \frac{B[R + (1+2\nu)a\cos\varphi] + a^2\cos^2\varphi}{a^2B^2} + \frac{h^2}{12}\Delta\left(\frac{B^2 + a^2\cos^2\varphi}{a^2B^2} + \Delta\right)$$

Dnepropetrovsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 191-197, November-December, 1971. Original article submitted June 1, 1971.

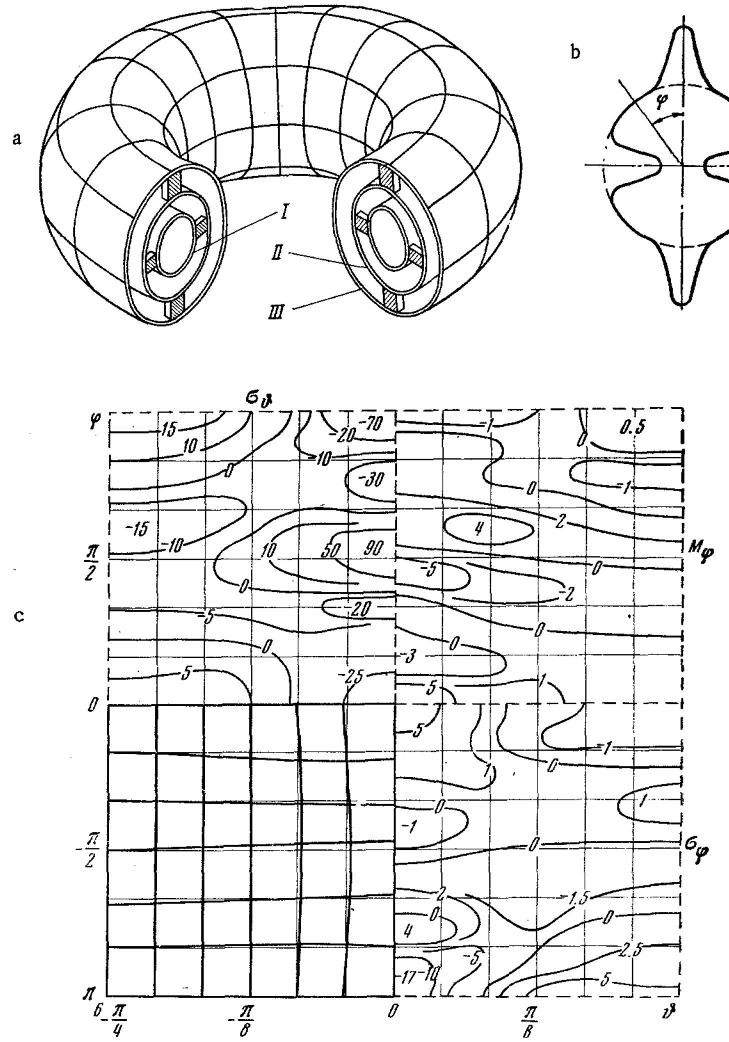

Fig. 1

Calculation results show that, for the concrete cases to be considered in the following, consideration of the operator A* alters the temperature loads by no more than 2%. Therefore (2) can be substituted approximately by the following:

$$A^\circ\left(\varphi, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\right) U(\varphi, \vartheta) = \Theta(\varphi, \vartheta) \tag{3}$$

The components of the vector $\Theta = (\Theta_1, \Theta_2, \Theta_3)$ of the temperature load are given by the expressions

$$\Theta_1 = -\frac{\varkappa Eh}{1-\nu}\frac{1}{a}\frac{\partial}{\partial\varphi}\left(t_1 + \frac{h}{6a}t_2\right)$$

$$\Theta_2 = -\frac{\varkappa Eh}{1-\nu}\frac{1}{B}\frac{\partial}{\partial\vartheta}\left(t_1 + \frac{h\cos\varphi}{6B}t_2\right)$$

$$\Theta_3 = 2\frac{\varkappa Eh}{1-\nu}\left(\frac{R + 2a\cos\varphi}{2aB}t_1 - \frac{h}{12}\Delta t_2\right)$$

Here ν, E, and h are Poisson's ratio, modulus of elasticity, and the thickness of the shell respectively; a is the radius of a meridional section of the shell; R is the distance between the center of this section and the axis of revolution; φ and ϑ are the curvilinear coordinates of the middle surface; $\varkappa$ is the coefficient of linear expansion of the material; t_1 and t_2 are the characteristics of the temperature field $T = T(\varphi, \vartheta)$ given by the integrals along the normal to the middle surface of the shell

$$t_1 = \frac{1}{h}\int_{-h/2}^{h/2} T\,d\gamma, \qquad t_2 = \frac{6}{h^2}\int_{-h/2}^{h/2} \gamma T\,d\gamma$$

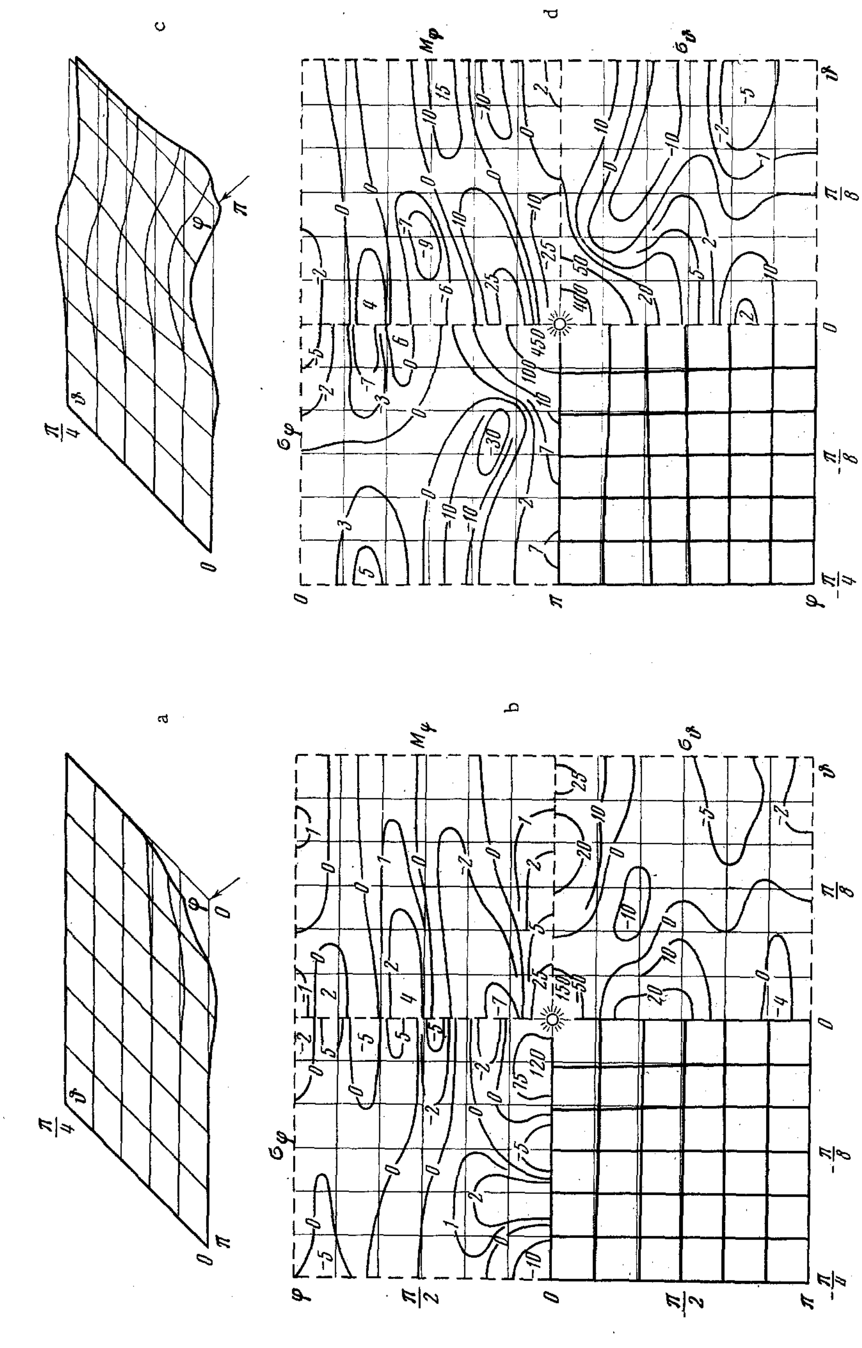

Fig. 2

TABLE 1

K	u_{k0}	u_{k2}	v_{k2}	w_{k0}	w_{k2}
0	0.00000	0.00000	−0.02560	0.00351	0.00274
1	−0.02898	−0.04245	−0.06262	0.02953	0.05439
2	−0.01887	−0.10094	0.06936	0.03799	0.19534
3	−0.00331	−0.06309	−0.02925	0.00969	−0.18607
4	0.00309	−0.00435	0.00493	−0.01237	0.01756
5	0.00085	−0.01354	0.00234	−0.00400	0.06725
6	−0.00084	0.00252	−0.00141	0.00519	−0.01602
7	−0.00011	0.00238	0.00017	0.00060	−0.00161

$B = R + a\cos\varphi$; Δ denotes the linear differential operators

$$\Delta = \frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{B^2}\frac{\partial^2}{\partial\vartheta^2} - \frac{\sin\varphi}{aB}\frac{\partial}{\partial\varphi}$$

If Green's matrix $G(\varphi, \vartheta; \alpha, \beta)$ of the system (3) for the given shell is known, then the displacement vector of its middle surface is given by the integral

$$U(\varphi, \vartheta) = \iint_{\Omega} G(\varphi, \vartheta; \alpha, \beta)\,\theta(\alpha, \beta)\, d_{\alpha,\beta}\Omega \tag{4}$$

The calculation of the thermoelastic state of the shell is normally carried out in stages:

1) determination of the temperature field of the shell;

2) computation of the temperature loads;

3) calculation of the stress-strain state of the shell under the action of these loads.

The complex nature of the temperature loads should be regarded as their special feature. The point is that the complications arising when the temperature fields are being determined are added to the complexities of the initial data, and then the complications of geometrical origin, appearing when the temperature loads are being computed, are added. It is obvious that to obtain trustworthy results we must at least take into account the basic ones of the factors just mentioned. The possibility of achieving this by means of Green's matrices, which have been calculated beforehand with sufficient accuracy, is illustrated by examples.

In the technology a construction is used which consists of three closed thin toroidal shells connected with one another by means of cylindrical branch pipes or bars, for example, according to the setup of Fig. 1a. The construction is in a medium with the temperature T_1, while inside the shell I a temperature T_0 is maintained. Thus, the thermal state of the shell II can approximately be characterized by local heating (cooling) in the neighborhoods of points (corresponding to the disposition of the bars) which are located sufficiently far from one another. Therefore, the problem concerned with the effect of local heating on the stress-strain state of a toroidal shell is of interest.

We consider the case when the temperature of the shell is given by the expression

$$T(\varphi, \vartheta) = T_* \cos^{16}{}^1\!/_2\varphi \cos^8 2\vartheta \qquad (T_* = \text{const})$$

(this corresponds to local heating in the neighborhoods of the four equally spaced points of the outer equator of the shell).

The approximation of the components of the vector Θ of the temperature load by the polynomials

$$\Theta_1(\varphi, \vartheta) \approx \sum_{k,m=0}^{7} \Theta_1^{km} \sin k\varphi \cos m\vartheta, \qquad \Theta_2(\varphi, \vartheta) \approx \sum_{k,m=0}^{7} \Theta_2^{km} \cos k\varphi \sin m\vartheta$$
$$\Theta_3(\varphi, \vartheta) \approx \sum_{k,m=0}^{7} \Theta_3^{km} \cos k\varphi \cos m\vartheta \tag{5}$$

guarantees in the given case a degree of accuracy exceeding 97%. Such smallness of error of the system (3) enables us to regard as sufficient the accuracy of Green's matrix used which completely takes into account the terms figuring in the expansions (5) (the computation error is insignificant).

The characteristics of the stress-strain state of the shell with the parameters $R=400$, $a=100$, $h=1$, $\nu=0.25$, calculated according to the expression (4), are shown in Fig. 2a, b. On the partial views c and d we have shown the results of calculating the same toroidal shell subjected to local heating in the neighborhoods of the four equally spaced points of the inner equator. In the upper parts of Fig. 2 we have shown the deflection of the shell; the points of heating are indicated by arrows. The tangential deformation of the coordinate grid is represented in the lower left corners of the partial views b and d. Here the normal stresses σ_φ and σ_ϑ, as well as the bending moment M_φ, are depicted by level lines.

The localization of perturbations within the limits of the Gaussian curvature of the middle surface having the same sign can be noted. This circumstance allows us to take into account local effects applied simultaneously at points of the outer and inner equators, by combining the cases considered.

The decrease of perturbations in the direction of the parallels, as we move away from the point of application of the concentrated effect in a region of positive Gaussian curvature, is expressed considerably more strongly than in a region of negative curvature. This means that the first of these regions of a toroidal shell has a greater stiffness with respect to the effects considered than the second region.

The case where shells are joined according to the scheme of Fig. 1a (the section $\vartheta=0$; the components $\Theta_3(\varphi, \vartheta)$ of the temperature load are shown in Fig. 1b, while the components $\Theta_1(\varphi, \vartheta)$ and $\Theta_2(\varphi, \vartheta)$ have a form corresponding to this) can also easily be calculated by this method. Certain results of such a calculation for a toroidal shell with the previous values of the parameters are shown in Fig. 1c.

The fact that the neighborhoods of the "cooling" bars are the ones with a maximum stress, and that the stresses σ_ϑ well exceed σ_φ, is of interest.

We next consider the problem of using an analogous method for taking into account temperature fields which vary more smoothly. Let, for example, in a medium of temperature $T=T_*x^2$ ($T_*=\text{const}$) there be a toroidal shell of radius $R=200$ and with the same values of the remaining parameters as above. The stationary temperature field of a shell obviously is given by the expression

$$T(\varphi, \vartheta) = T_*(R + a\cos\varphi)^2\cos^2\vartheta$$

The displacement components of the middle surface in this case have the form

$$u(\varphi, \vartheta) = \sum_{k=0}^{\infty}(u_{k0} + u_{k2}\cos 2\vartheta)\sin k\varphi$$

$$v(\varphi, \vartheta) = \sum_{k=0}^{\infty} v_{k2}\sin 2\vartheta\cos k\varphi$$

$$w(\varphi, \vartheta) = \sum_{k=0}^{\infty}(w_{k0} + w_{k2}\cos 2\vartheta)\cos k\varphi$$

The values of the coefficients u_{km}, v_{km}, w_{km} are presented in Table 1. Without complicating the discussion, we note only that a rapid decrease of the moduli of the Fourier coefficients of displacements ensures that acceptable accuracy is achieved while calculating the characteristics of the stress state of the shell.

It is easy to see that in principle a possibility exists for extending the calculation schemes presented here to any shell for which Green's matrices can be calculated beforehand according to [1, 2].

LITERATURE CITED

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